Fluid Dynamics:

Conservation of Mass, Energy and Momentum

Objectives

- **Introduce concepts necessary to analyze fluids in motion**
- **Identify differences between Steady/unsteady uniform/non-uniform compressible/incompressible flow**
- **Introduce conservation of mass principle**
  - Continuity equation
  - Steady state mass balance
  - Unsteady state mass balance
- **Introduce Conservation of energy principle**
  - First Law of Thermodynamics
  - Derive Bernoulli's equation: practical applications
  - Steady state energy balance
  - Application: Fluid flow measurement
- **Introduce conservation of momentum principle**
  - Momentum transfer: the concept of balance of forces.

Learning Outcome

At the end of the chapter, you should be able to:

1. Define continuity equation to solve steady state and unsteady state mass balance.
2. Define Bernoulli's equation.
3. Apply the energy balance to determine flow characteristics of fluid in various devices such as orifice meter, venturi meter and pitot tube.
4. Apply the concept of momentum balance to determine the forces and pressure in fluid flow problems.

Fluid dynamics

The study of the effect forces on fluid motion. The motion of fluids can be predicted in the same way as the motion of solids are predicted using the fundamental laws of physics together with the physical properties of the fluid.

It is not difficult to envisage a very complex fluid flow. waves on beaches; hurricane sand tornadoes or any other atmospheric phenomenon are all example of highly complex fluid flows which can be analyzed with varying degrees of success (in some cases hardly at all!). There are many common situations which are easily analyzed.

Various fluid flow

- **Uniform flow**
  - If velocity at a given instant is the same in magnitude and direction at every point in the fluid.
- **Non-uniform flow**
  - Velocity changes from point to point

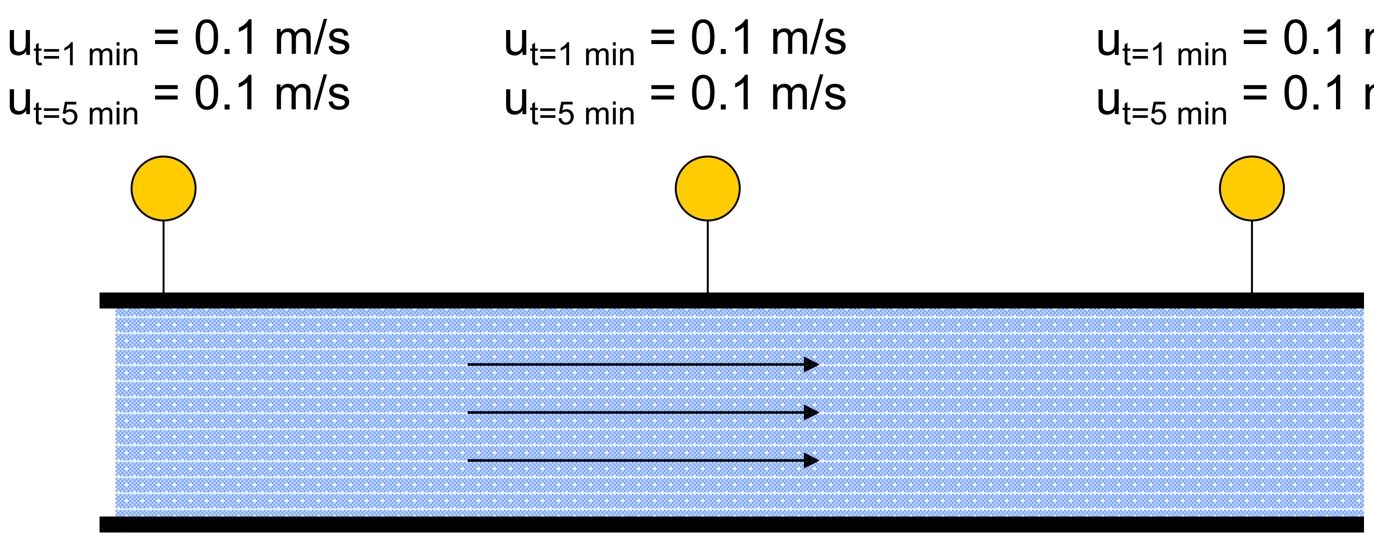
- **Steady flow**:
  - Velocity, pressure and cross section of the fluid stream may vary from point to point, but **do not** change with time.
- **Unsteady flow**:
  - Conditions do change with time

Combining the above we can classify any flow into one of four type

1. Steady uniform flow
2. Steady non-uniform flow
3. Unsteady uniform flow
4. Unsteady non-uniform

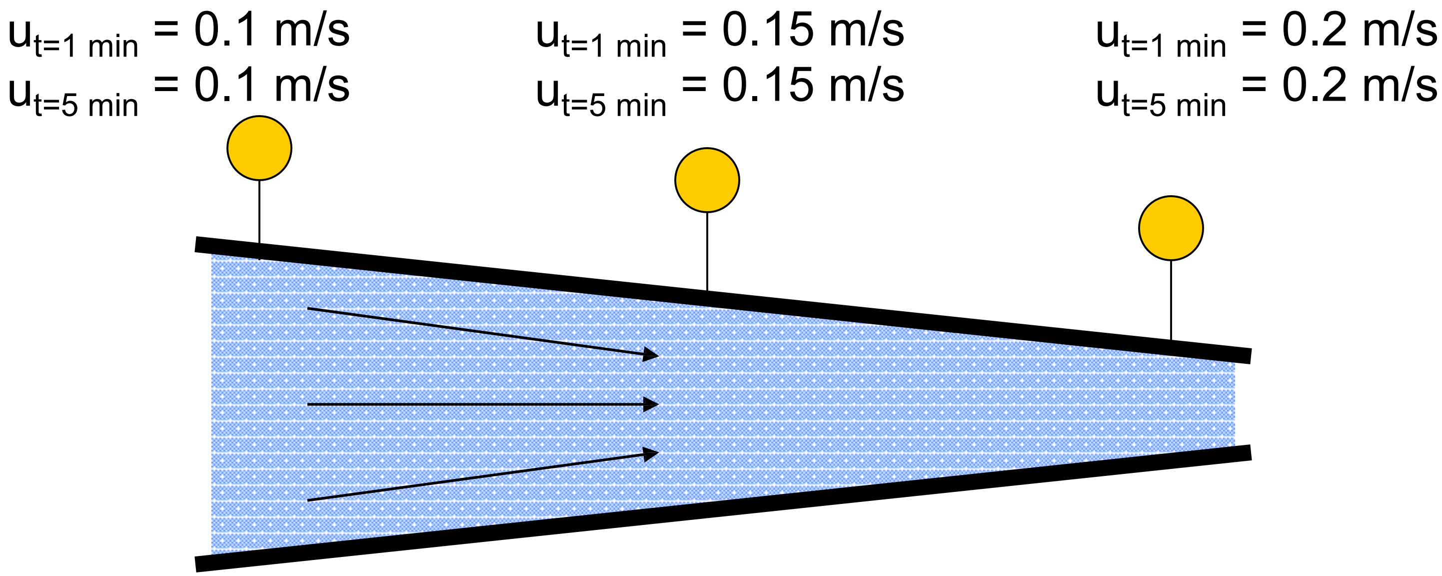
Steady, uniform flow:

- Conditions do not change with position or time.
- E.g.: flow of fluid through pipe of uniform bore (cross section) running completely full.



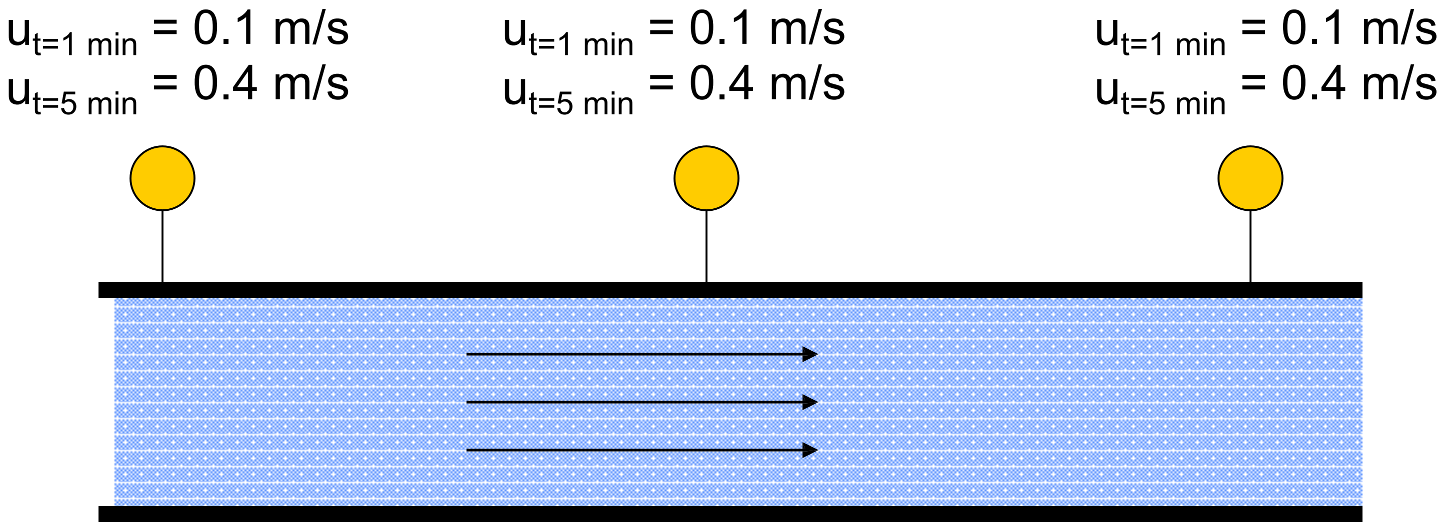
Steady, non-uniform flow:

- Conditions change from point to point, but not with time.
- E.g.: Flow of liquid at a constant rate through a tapering pipe running completely full.



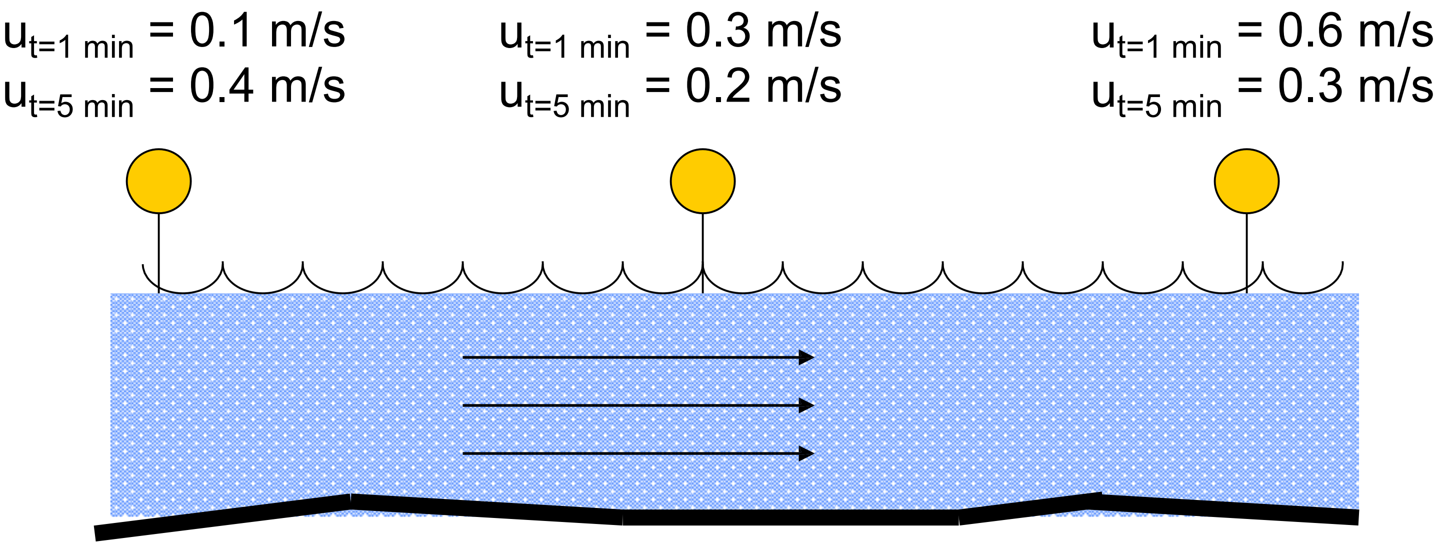
Unsteady, uniform flow:

- At a given instant of time, velocity at every point is the same, but this velocity will change with time.
- E.g.: accelerating flow of liquid through a pipe of uniform bore running full (during pump start-up)



Unsteady, non-uniform flow:

- Cross sectional area and velocity may vary from point to point, and also change with time.
- E.g.: a wave traveling along a channel.



Among these four types of flow which one is the most simple and why?

Flow Rate

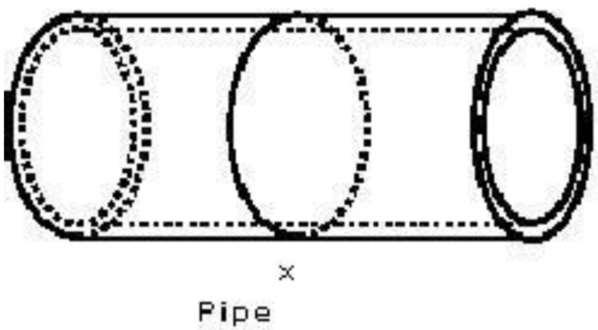
Mass flow rate

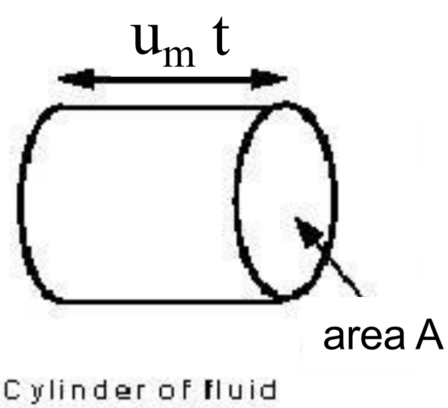
$$\text{Mass flow rate} = \dot{m} = \frac{mass\ of\ fluid\ in\ the\ bucket}{time\ taken\ to\ collect\ the\ fluid}$$

Volume flow rate (some times called discharge)

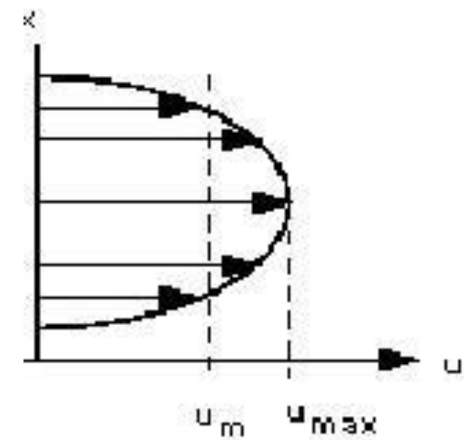
Volume flow rate= Q =Volume of fluid/time
= mass of fluid/(density x time)
= mass flow rate/ density

Relation between discharge and mean velocity





Here, u_m is mean velocity



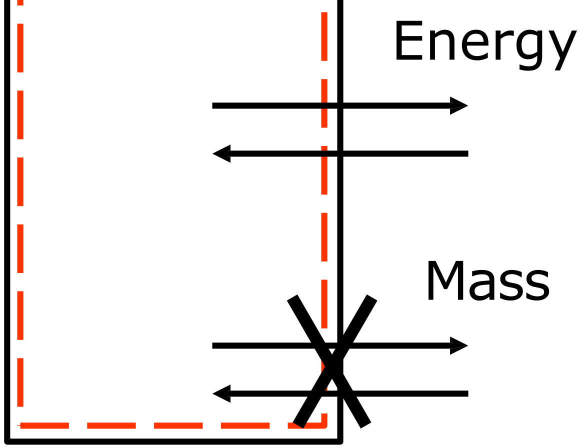
A typical velocity profile across a pipe

$$Q = \text{volume/time} = (u_m \times t \times A)/t$$
$$= u_m A$$

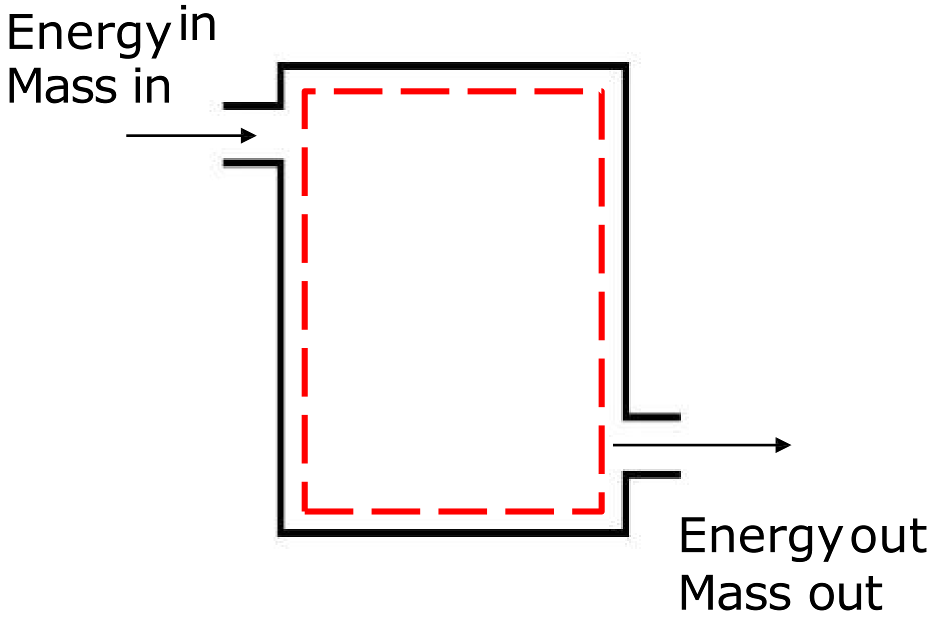
if the cross-section area, A, is $1.2 \times 10^{-3}\ m^2$ and the volume rate is 24 l/s, What is the mean velocity ?

Two different systems

- **Closed system**
  - Control mass
  - Only energy is transferred the boundary, but not mass



- **Open system**
  - Control volume
  - Both mass and energy is transferred the boundary



General Balance Equation

Flow in – Flow out + Creation – Destruction = Accumulation

In Material & Energy Balance

Input – Output + Generation – Consumption = Accumulation

In Non-reacting systems

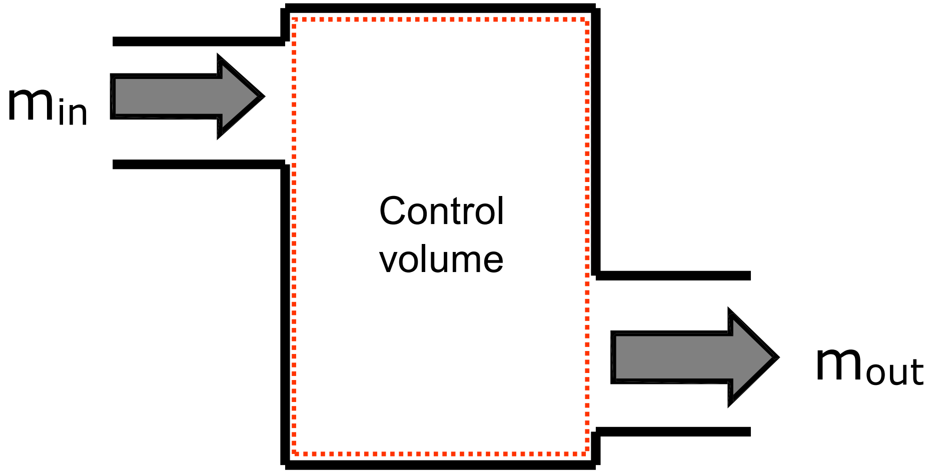
Input – Output = Accumulation

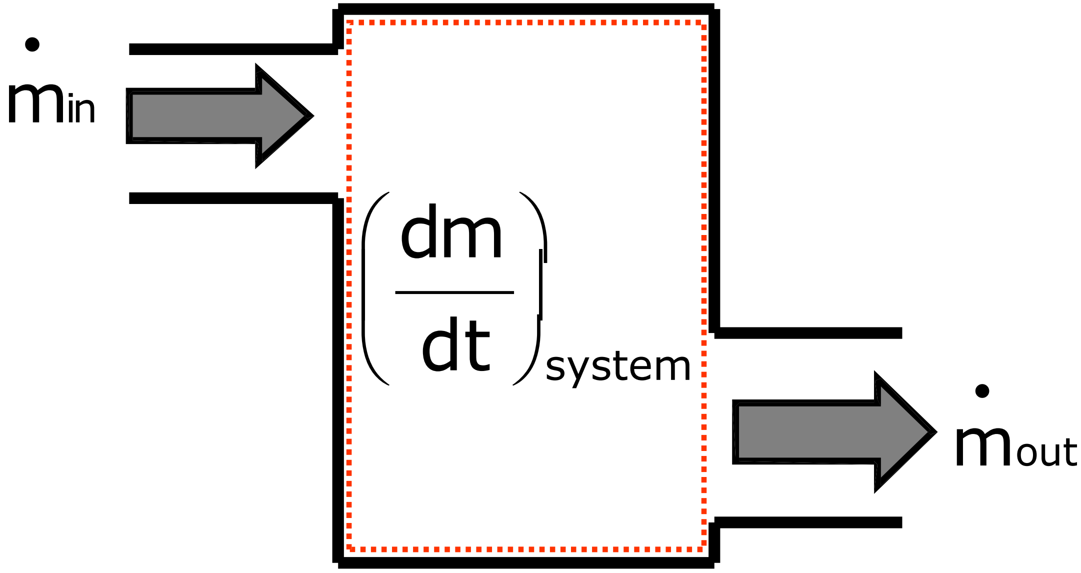
Criteria of balance equation

- Must apply to some period of time.
- Balance equation deals with changes in the thing being accounted for.
- Applied only to any countable set of unit (quantitative)
  (EXTENSIVE PROPERTY – e.g mass, energy, momentum)
- Not applied for uncountable set of unit (qualitative)
  (INTENSIVE PROPERTY – e.g T, P, color)

Continuity (Conservation of Mass)

Matter cannot be created or destroyed - (it is simply changed in to a different form of matter). This principle is know as the conservation of mass and we use it in the analysis of flowing fluids.





$$\left(\frac{dm}{dt}\right)_{system} = \dot{m}_{in} - \dot{m}_{out}$$

For steady state flow, there is no increase in the mass with in the control volume/chosen boundary, means $(dm/dt)_{system} = 0$

Steady state mass balance

(No accumulation)

$$\left(\frac{dm}{dt}\right)_{system} = 0$$

$$\dot{m}_{in} = \dot{m}_{out}$$

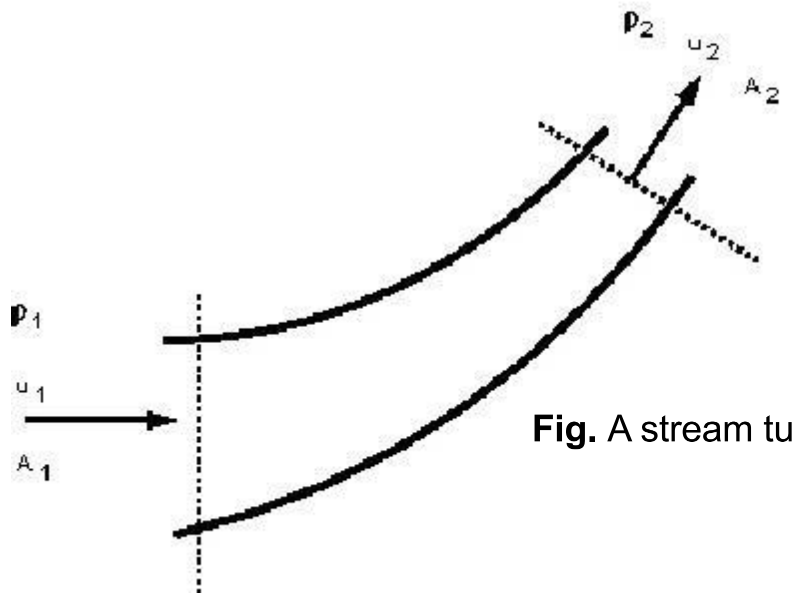


Fig. A stream tube

We can write, $\rho_1 u_1 A_1 = \rho_2 u_2 A_2$

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 = \text{constant} = \text{mass flow rate}$$

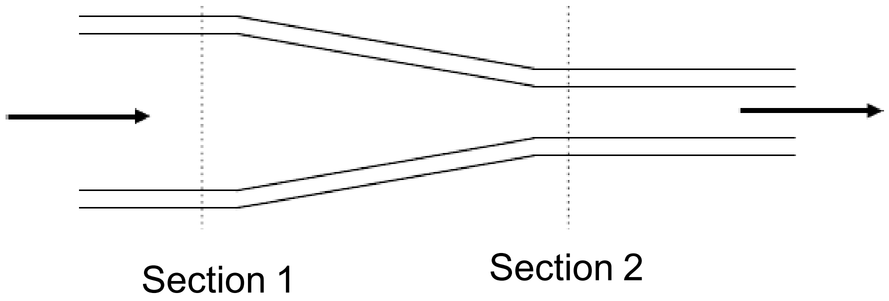
This is the equation of continuity
(**mass flow rate is the same in each section**)

For incompressible fluid *e.g.*, does not change $\rho_1 = \rho_2 = \rho$

$$u_1 A_1 = u_2 A_2 = Q$$

This is the form of continuity equation most often used

Some examples of continuity equation applications

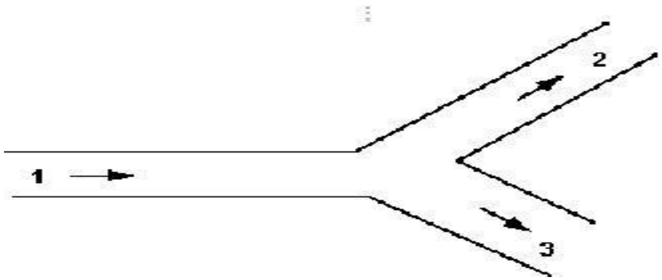


Discharge at Section 1 = discharge at section 2

$\rho_1 Q_1 = \rho_2 \ Q_2$ (if consider incompressible fluid)

so, $A_1 u_1 = A_2 u_2$

$u_1 d_1^2 = u_2 d_2^2$



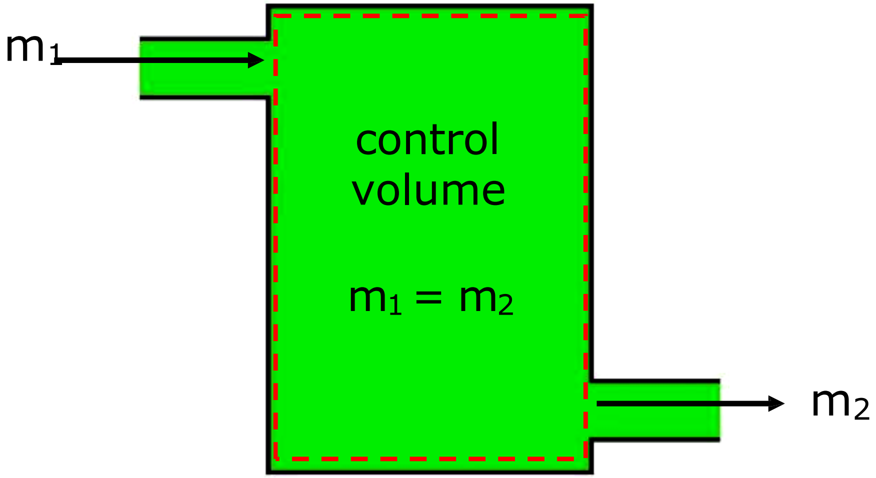
Total mass flow into the junction = Total mass out of the junction

$\rho_1 Q_1 = \rho_2 Q_2 + \rho_3 Q_3$

Incase of incompressible fluid, $A_1 u_1 = A_2 u_2 + A_2 u_2$

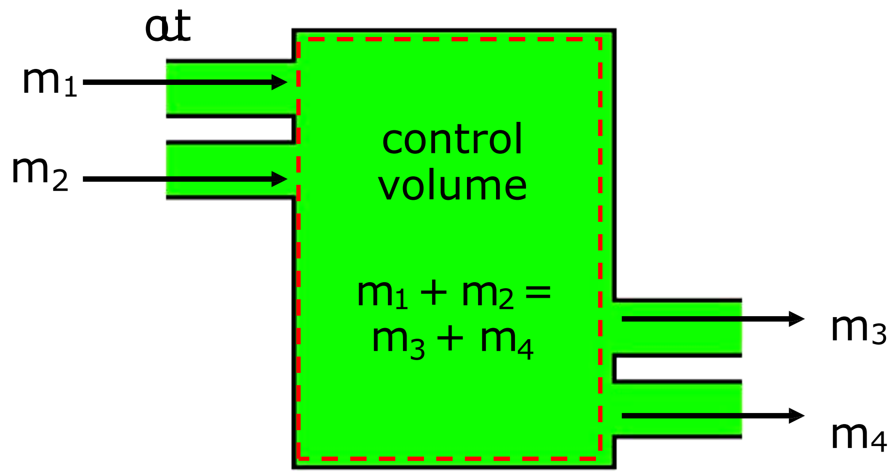
For single stream in and out,

$$m_1 = m_2$$



For multiple streams,

$$\sum m_{in} = \sum m$$

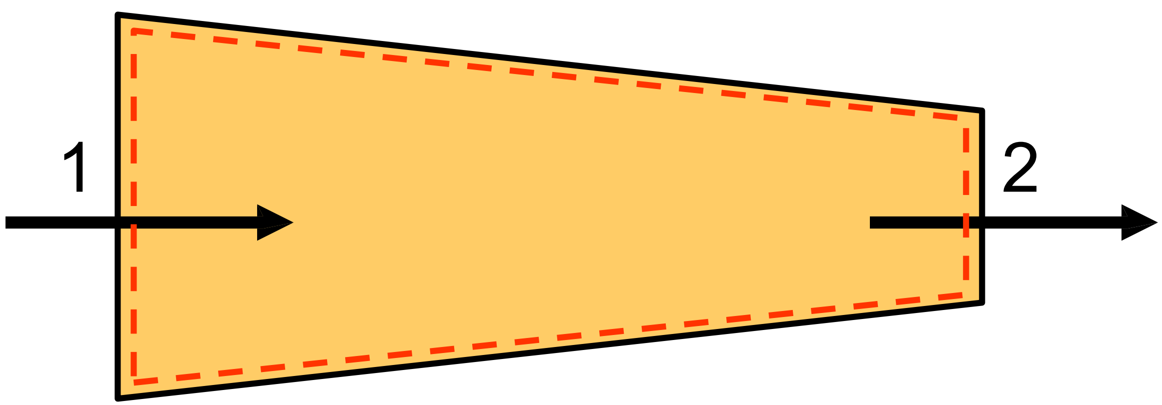


Unsteady state mass balance

$$\left(\frac{dm}{dt}\right)\Bigg|_{system} = \dot{m}_{in} - \dot{m}_{out}$$

Class example 1

- Water is flowing in a pipe. At point 1 the inside diameter is 0.25 m and the velocity is 2 m/s. Determine:
  - the mass flow rate
  - volumetric flow rate?
  - the velocity at point 2 where the inside diameter is 0.125 m?



Solution

- Assumption:
  - Steady flow system,
  - density is constant = 1000 kg/m³

a) Mass flow rate $\dot{m} = \rho A u = \left(1000\,\frac{kg}{m^3}\right)\left(\frac{\pi}{4}(0.25m)^2\right)\left(2\,\frac{m}{s}\right)$

$$= 98.2\,\frac{kg}{s}$$

b) Volumetric flow rate

$$Q = \frac{\dot{m}}{\rho} = \frac{98.2\,\frac{kg}{s}}{1000\,\frac{kg}{m^3}} = 0.0982\,\frac{m^3}{s}$$

Solution

a) From continuity equation

$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 \rightarrow A_1 u_1 = A_2 u_2$ (since density is constant)

$$\left(\frac{\pi}{4} D_1^2\right) V_1 = \left(\frac{\pi}{4} D_2^2\right) V_2$$

$$\Rightarrow V_2 = V_1 \frac{D_1^2}{D_2^2}$$

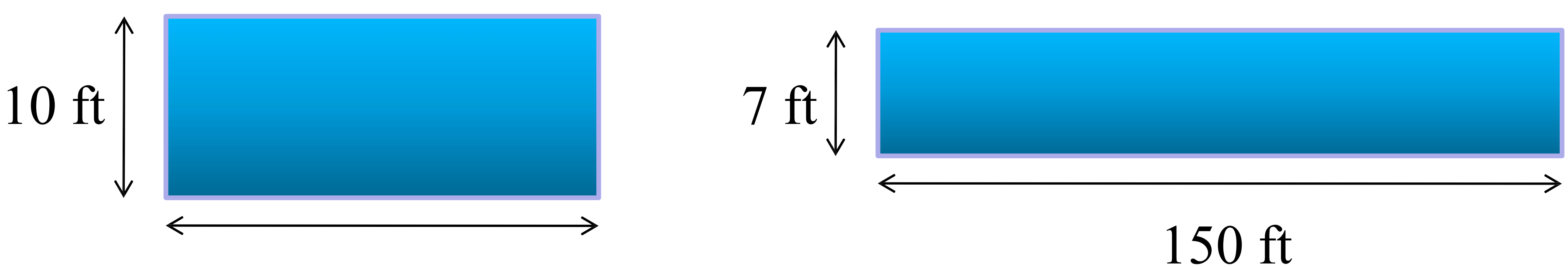
$$= \left(2\frac{m}{s}\right)\left(\frac{0.25^2}{0.125^2}\right)$$

$$= 8\frac{m}{s}$$

Class example 2

A river has a cross section that is approximately a rectangle with 10 ft deep and 50 ft wide. The average velocity is 1 ft/s. How many U.S. gallons per minute pass a given point? What is the average velocity at a point downstream, where the channel shape has changed to 7 ft in depth and 150 ft in wide?

Cross sectional area of the river:



Solution

- Assumption: Steady flow, ρ_{H2O} is constant = 62.4 Ib_m/ft^3

- Volumetric flow rate,

$$Q = A_1 u_1 = (10\ \text{ft})(50\ \text{ft})\left(1\ \frac{\text{ft}}{\text{s}}\right) = 500\ \frac{\text{ft}^3}{\text{s}}$$

$$= 500\ \frac{\text{ft}^3}{\text{s}} \times \frac{60\ \text{s}}{\text{min}} \times \frac{7.48\ \text{U.S gal}}{\text{ft}^3}$$

$$= 224400\ \frac{\text{U.S gal}}{\text{min}}$$

Solution

- Velocity at downstream:

$$Q = 500 \frac{ft^3}{s} = A_2 u_2$$

$$u_2 = \frac{500 \frac{ft^3}{s}}{(7\ ft)(150\ ft)}$$

$$= 0.476 \frac{ft}{s}$$

Class Example 3

- A lake has a surface area of 100 km^2.

  One river is bringing water into the lake at a rate of 10,000 m^3/s, while another is taking water out at 8000 m^3/s.

  Neglecting evaporation of the water, how fast is the level of the lake rising or falling?

Solution

Assumption: Steady flow, density of the lake water is constant, evaporation rate is negligible.

$$\frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

$$\frac{d(\rho V)}{dt} = \rho Q_{in} - \rho Q_{out}$$

$$\frac{dV}{dt} = Q_{in} - Q_{out}$$

$$\frac{d(Ah)}{dt} = Q_{in} - Q_{out}$$

$$\frac{dh}{dt} = \frac{Q_{in} - Q_{out}}{A}$$

$$= \frac{10000\frac{m^3}{s} - 8000\frac{m^3}{s}}{100\ km^2 \times \frac{1000^2\ m^2}{km^2}}$$

$$= 2 \times 10^{-5}\frac{m}{s}\left(\text{or } 0.02\frac{mm}{s}\right)$$

Conservation of Energy

Objectives

- Concept of First Law of Thermodynamics
- Derive Bernoulli’s equation
- Application of Bernoulli’s equation
- General energy balance equation

Learning Outcomes

At the of this chapter you should be able to:

- Express Bernoulli’s equation
- Simplify Bernoulli’s equation for various cases.
- Solve problems using Bernoulli’s equation.

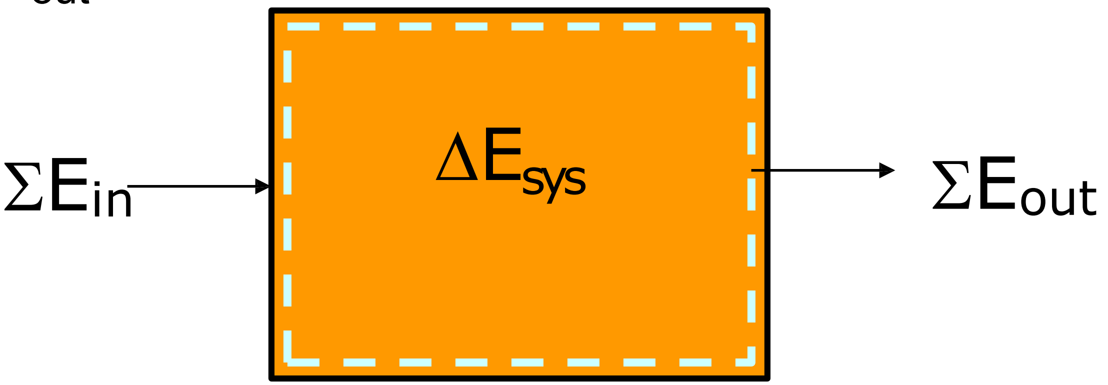
The First Law of Thermodynamics

(The conservation of energy principle)

Energy cannot be created nor destroyed, but can be transferred from one form to another.

Accumulation = Flow in – Flow out

$$\Delta E_{sys} = E_{in} - E_{out}$$



For steady state:

$$\mathbf{E_{in} = E_{out}}$$

Forms of energy

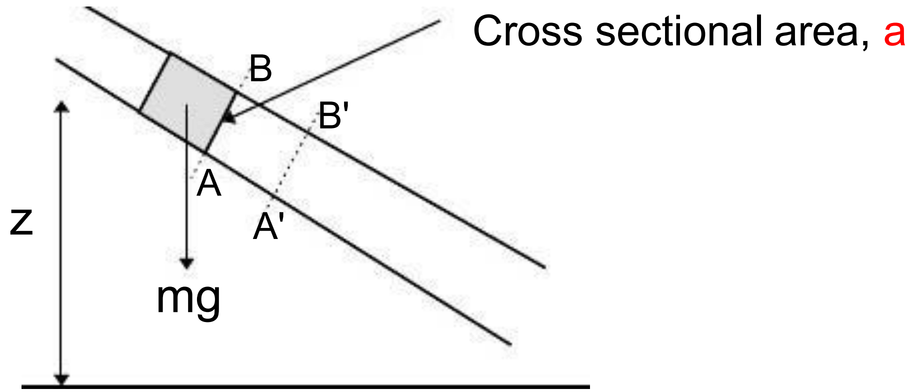
1. Potential energy 2. Kinetic energy 3. Flow energy/internal energy 4. Electrostatic 5. Magnetic 6. Nuclear

Derivation of Bernoulli's equation

Bernoulli's equation has some restrictions in its applicability, they are:

- Flow is steady;
- Density is constant (which also means the fluid is incompressible);
- Friction losses are negligible.
- The equation relates the states at two points along a single streamline, (not conditions on two different streamlines).

Derivation of Bernoulli's equation



An element of fluid, as that in the Fig., has potential energy due to its heights z above a datum and kinetic energy due to its velocity V. If the weight is mg then,

Potential energy (PE) = mgz

PE per unit weight = z

Kinetic energy (KE)= $mV^2/2$

KE per unit weight = $V^2/2g$

Flow energy : Represents the amount of work necessary to move the element of fluid across a certain section against the pressure, P.

At any cross-section the pressure generates a force, the fluid will flow, moving the cross-section, so work will be done. If the pressure at cross section AB is P and the area of the cross-section is a, then

Force on AB = Pa

when the mass **mg** of fluid has passed AA', cross-section AB will have moved to A'B'

$$\text{volume passing AB} = \frac{mg}{\rho g} = \frac{m}{\rho}$$

Therefore,

$$\text{distance A'B'} = \frac{m}{\rho a}$$

$$\text{Work done} = \text{force x distance A'B'} = \text{Pa x } \frac{m}{\rho a} = \frac{P}{\rho}m$$

$$\text{Work done per unit weight} = \frac{P}{\rho g}$$

This term also known as pressure energy in flowing system

Summing all of these energy terms gives

Pressure energy per unit weight + Kinetic energy per unit weight + Potential energy per unit weight = Total energy per unit weight

or,

$$\frac{P}{\rho g} + \frac{u^2}{2g} + z = H$$

Bernoulli's equation is one of the most important/useful equations in fluid mechanics

As all of these elements of the equation have units of length, they are often referred to as the following:

$$\frac{P}{\rho g} + z + \frac{u^2}{2g} = H$$

Pressure head

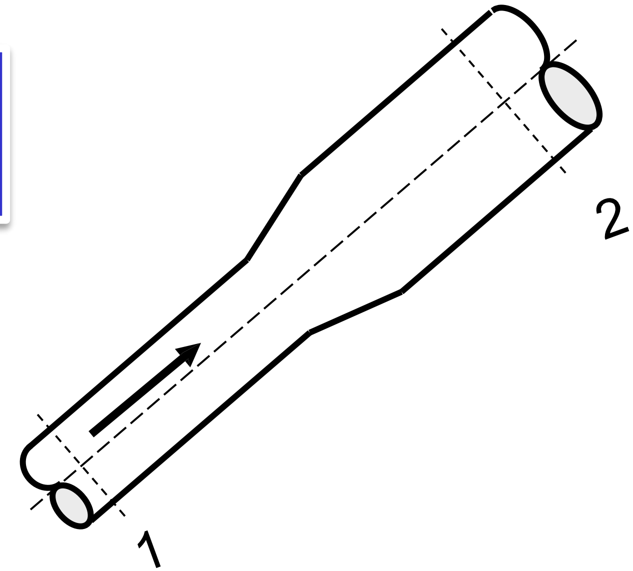
Elevation head / Gravity head

Velocity head

By the principle of conservation of energy the total energy in the system does not change, Thus the total head does not change. So the Bernoulli equation can be written

$$\frac{P}{\rho g} + \frac{u^2}{2g} + z = H = \text{constant}$$

We can apply it between two points, 1 and 2, on the streamline in the figure



total energy per unit weight at 1 = total energy per unit weight at 2

or total head at 1 = total head at 2

or $$\frac{P_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

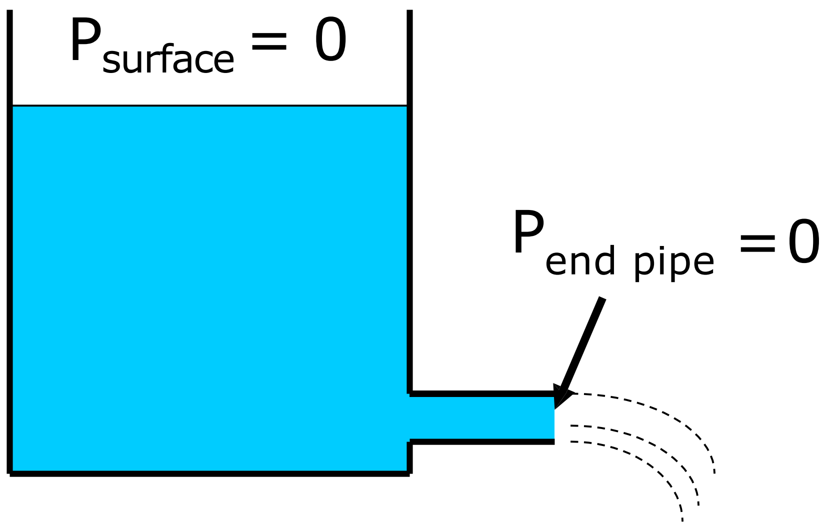
or $$\frac{\Delta P}{\rho} + g\Delta z + \frac{\Delta V^2}{2} = 0$$

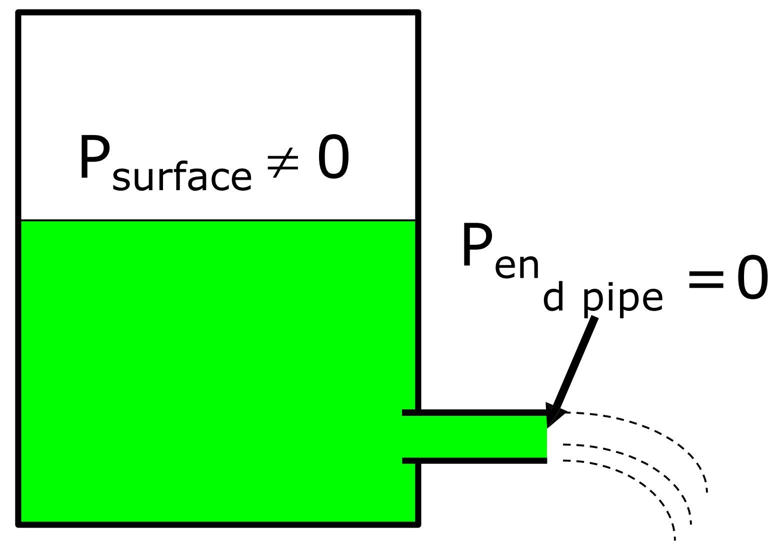
Solving problems using Bernoulli's equation

1. Decide the items are known and what is to be determined.
2. Decide the two sections in the system will be used when writing Bernoulli's equation.
3. Write Bernoulli's equation for the two selected sections in the system.
    - The equation should be written in the direction of flow
4. Made clear when labeling the terms in the equation
    - Note the reference points on the sketch of the system
5. Simplify the equation by canceling terms that are zero or equal on both sides of the equation.
6. Solve the problem!

Important Points

1. When the fluid at a reference point is exposed to the atmosphere, the pressure is zero gage pressure.
   - The pressure term (thus pressure head) is cancelled from the equation.



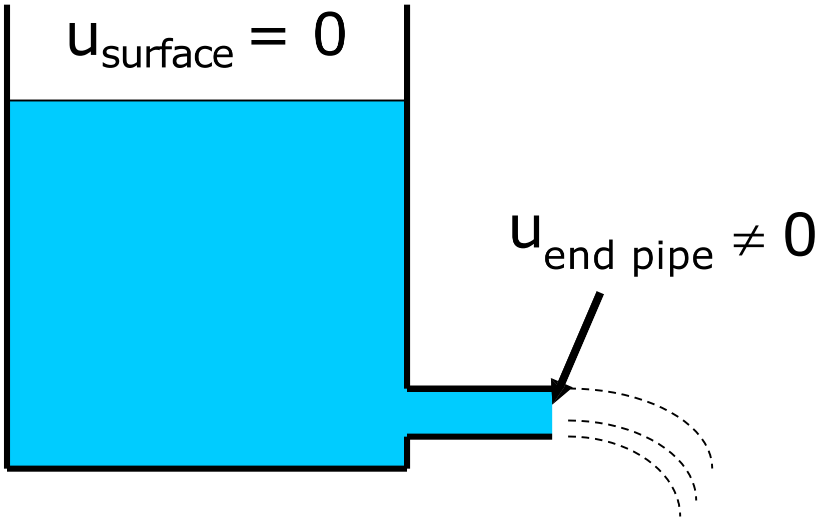


$$\cancelto{0}{\frac{\Delta P}{\rho}} + g\Delta z + \frac{\Delta u^2}{2} = 0$$

Important Points

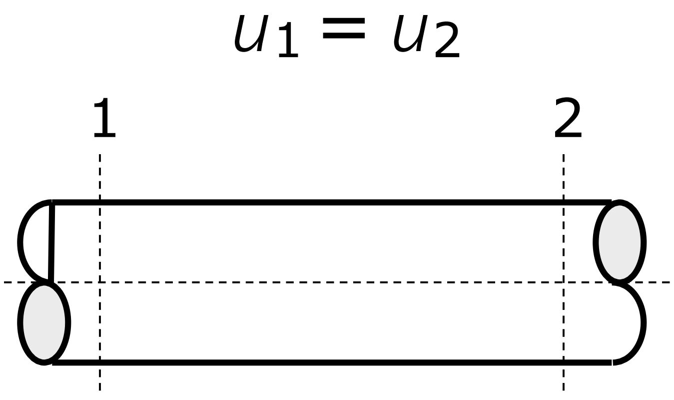
2. The velocity at the surface of a tank or a reservoir is considered to be zero.
   - The velocity (thus velocity head) term can be cancelled in the equation.
   - Applicable if ratio of cross sectional area of two reference points are very small,

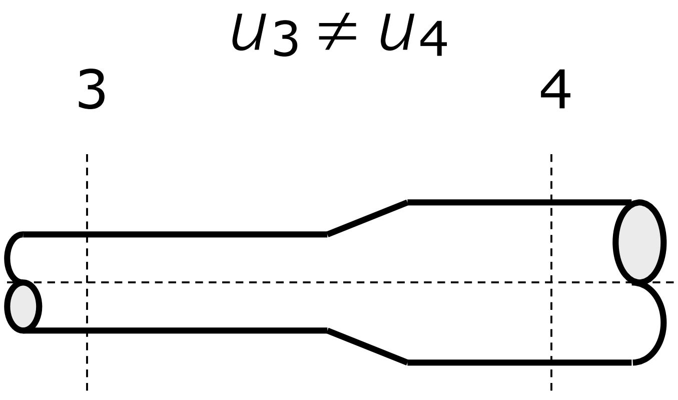
$$\frac{d_1}{d_2} \cong 0$$



Important Points

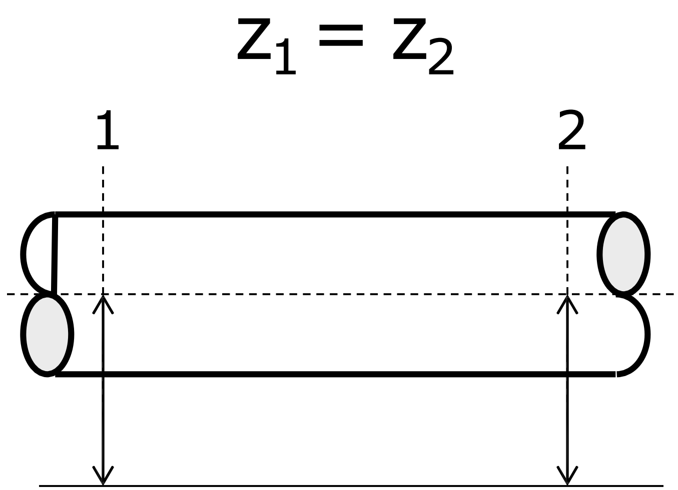
3. When both reference points are in the same pipe (same cross sectional area),
   - The velocity (thus velocity head) term can be cancelled in the equation.

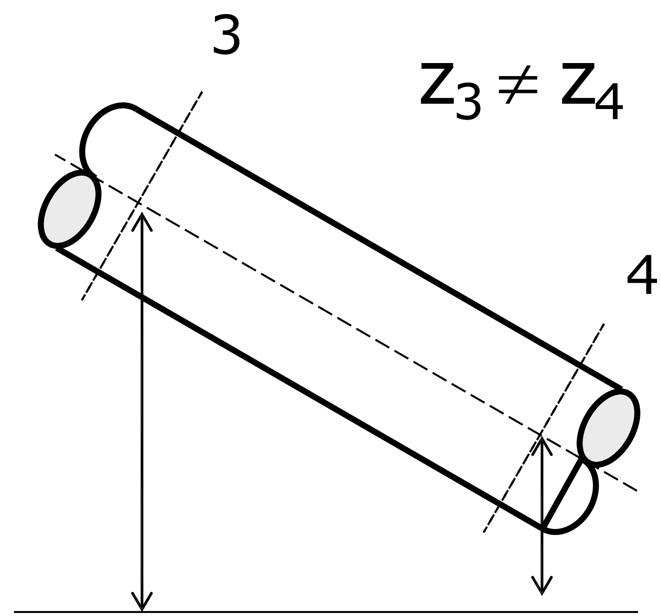




Important Points

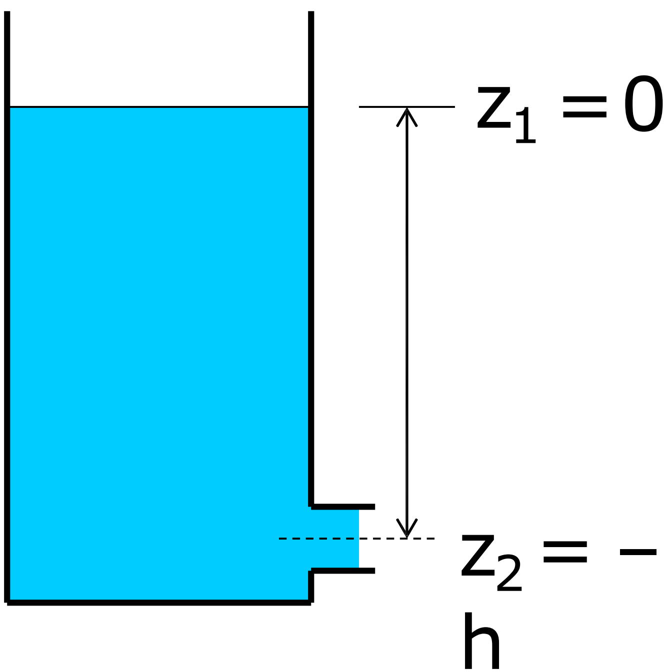
4. When the two points of reference are at the same elevation,
   - The elevation (thus gravity head) term can be canceled.

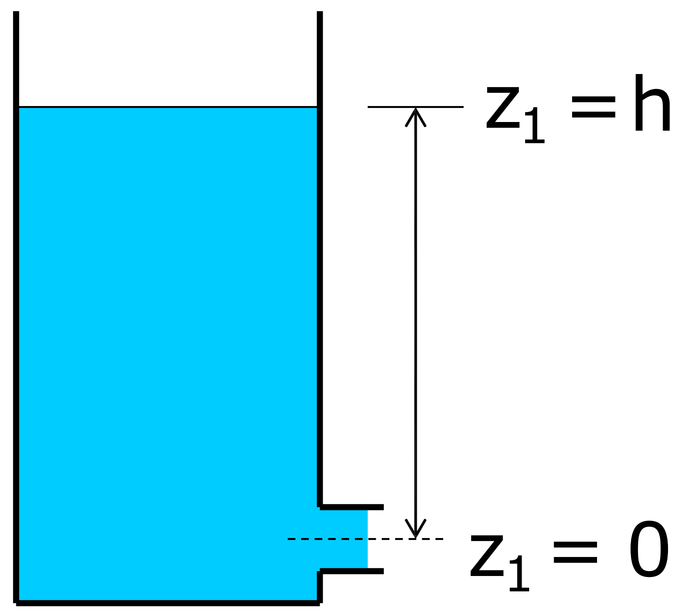




Important Points

5. When referring to elevation, beware of the 'sign'.
   - By fixing one reference point as 'datum', give appropriate sign to the elevation



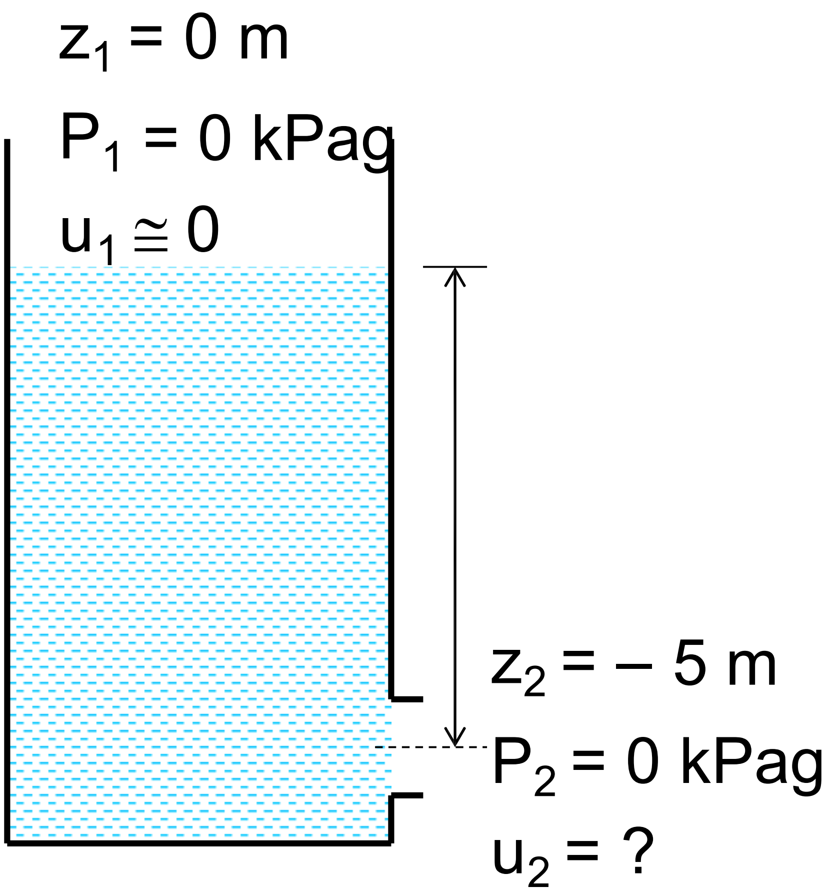


Class Example 1

A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap. A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

Solution:

- Assumption:
  - No work interaction.
  - The rounded outlet is frictionless
  - Tank is very large that the velocity of water in tank is relatively small, $u_1 \cong 0$
  - Steady flow



$$\frac{\Delta P}{\rho} + g\Delta z + \frac{\Delta u^2}{2} = 0$$

$$g(z_2 - z_1) + \frac{u_2^2 - u_1^2}{2} = 0$$

$$u_2 = \sqrt{2g(z_1 - z_2)}$$

Toricelli's equation

$$= \sqrt{2(9.81)(0 - (-5))}$$

$$= 9.9 \frac{\text{m}}{\text{s}}$$