Pertemuan 15 Kalkulus 3

SOAL DAN JAWABAN MATERI UJIAN AKHIR SEMESTER GANJIL 2020/2021

1. Tentukan deret kuasa dalam $(x-a)$ dari

   a. $f(x) = sin\,2\,x$ dengan $a = \frac{\pi}{2}$
   b. $f(x) = e^{-3x}$ dengan $a = 0$

2. Tentukan deret fourier dari fungsi periodik

$$f(x) = \begin{cases} -1 & ; -2 \le x < 0 \\ 1 & ; \quad 0 \le x < \quad 2 \end{cases} \quad \text{dengan periode P} = 4$$

Jawaban :

1.a. Tentukan deret kuasa dalam $(x-a)$ dari $f(x) = sin\,2\,x$ dengan $a = \frac{\pi}{2}$

$f(x) = sin\,2\,x$ $\qquad f\left(\frac{\pi}{2}\right) = \sin 2.\frac{\pi}{2} = \sin\pi = 0$

$f'(x) = 2\,cos\,2x$ $\qquad f'\left(\frac{\pi}{2}\right) = 2\,cos\,2\frac{\pi}{2} = 2\cos\pi = 2.-1 = -2$

$f''(x) = 2.-2\,sin\,2x = -2^2\sin 2x$ $\qquad f''\left(\frac{\pi}{2}\right) = -2^2\sin 2\frac{\pi}{2} = -2^2\sin\pi = 0$

$f'''(x) = -2^2 . 2\cos 2x = -2^3 . \cos 2x$ $\qquad f'''\left(\frac{\pi}{2}\right) = -2^3\,cos\,2\frac{\pi}{2} = -2^3\cos\pi = -2^3.-1 = 2^3$

$f^{(4)}(x) = -2^3 . -2\sin 2x = 2^4 . \sin 2x$ $\qquad f^{(4)}\left(x\frac{\pi}{2}\right) = -2^4\sin 2\frac{\pi}{2} = -2^4\sin\pi = 0$

$f^{(5)}(x) = 2^5\,cos\,2x$ $\qquad f^{(5)}\left(\frac{\pi}{2}\right) = 2^5\,cos\,2\frac{\pi}{2} = 2^5\cos\pi = 2^5.-1 = -2^5$

$f^{(6)}(x) = -2^6\,sin\,2x$ $\qquad f^{(6)}\left(\frac{\pi}{2}\right) = 0$

Jadi dengan memasukan harga-harga ini ke deret Taylor diperoleh,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

$$sin\,2x = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x-\frac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\frac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\frac{\pi}{2})^3 + \ldots + \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x-\frac{\pi}{2})^n$$

$$sin\,2x = 0 - 2\left(x-\frac{\pi}{2}\right) + 0 + 2^3\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 + 0 - 2^5\frac{1}{5!}(x-\pi)^5 + \ldots$$

$$sin\,2x = -2\left(x-\frac{\pi}{2}\right) + 2^3\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - 2^5\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \ldots$$

$$= \sum_{n=1}^{\infty}(-1)^n . 2^{2n-1}\frac{1}{(2n-1)!}\left(x-\frac{\pi}{2}\right)^{2n-1}$$

1.b. Tentukan deret kuasa dalam $(x-a)$ dari $f(x)=e^{-3x}$ dengan $a=0$

$$
\begin{array}{ll}
f(x) = e^{-3x} & f(0) = e^{-3.0} = e^{0} = 1 \\
f'(x) = -3\,e^{-3x} & f'(0) = -3.\,e^{-3.0} = -3.\,e^{.0} = -3.1 = -3. \\
f''(x) = -3.-3\,e^{-3x} = 3^2\,e^{-3x} & f''(0) = 3^2\,e^{-3.0} = 3^2 \\
f'''(x) = 3^2.-3\,e^{-3x} = -3^3 e^{-3x} & f'''(0) = -3^3 e^{-3.0} = -3^3 \\
f^{(4)}(x) = -3^3\,.-3\,e^{-3x} = 3^4 e^{-3x} & f^{(4)}(0) = 3^4 e^{-3.0} = 3^4
\end{array}
$$

Jadi dengan memasukan harga-harga ini ke deret Mac Laurin $a=0$ diperoleh,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

$$e^{-3x} = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n$$

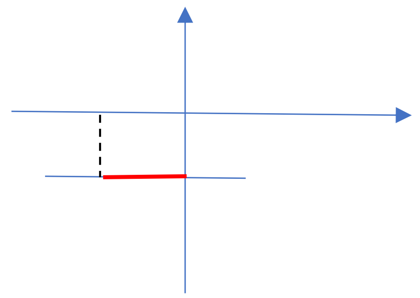
$$e^{-3x} = 1 - 3x + \frac{3^2}{2!}x^2 - \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 \ldots$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}.3^{n-1}\frac{1}{(n-1)!}x^{n-1}$$

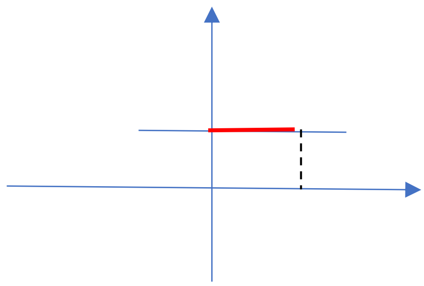
2. Tentukan deret fourier dari fungsi periodik

$$f(x) = \begin{cases} -1 & ; -2 \le x < 0 \\ 1 & ; \quad 0 \le x < \quad 2 \end{cases}$$ dengan periode P $=4=2L$ maka $L=2$

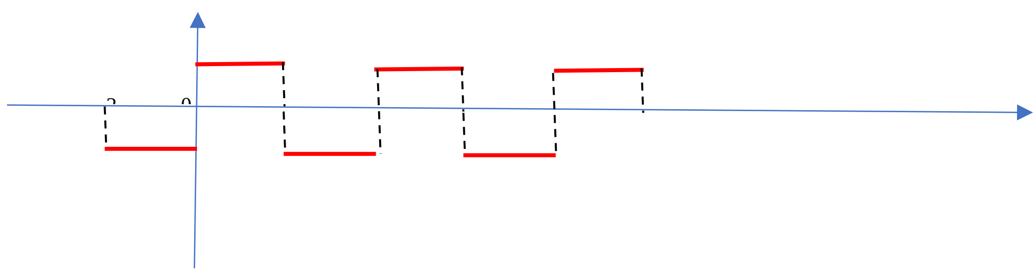
$y=f(x)=-1;\ \ -2 \le x < 0$



$y=f(x)=1;\ \ 0 \le x < 2$



Fungsi periodiknya adalah menggabungkan fungsi diatas



Fungsi periodik ganjil dengan periode p = 4 (-2<x<2) dan L=2

Dari modul 14 maka deret Fouriernya

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

Dengan

$$b_n = \frac{2}{L}\int_0^L f(x)\, f(x) sin\frac{n\pi}{L}x\, dx$$

maka

$$b_n = \frac{2}{2}\int_0^2 f(x)\, f(x) sin\frac{n\pi}{2}x\, dx$$

$$= \int_0^2 f(x)\, f(x) sin\frac{n\pi}{2}x\, dx \quad \text{karena} \quad f(x) = 1;\;\; 0 \le x < 2$$

$$= \int_0^2 1\, .\, sin\frac{n\pi}{2}x\, dx = \int_0^2 sin\frac{n\pi}{2}x\;\, dx$$

$$= \frac{2}{n\pi}.-cos\frac{n\pi}{2}x\, \Big|_0^2 = -\frac{2}{n\pi}\left(cos\frac{n\pi}{2}2 - cos\frac{n\pi}{2}0\right)$$

$$= -\frac{2}{n\pi}(\cos n\pi - \cos 0) = -\frac{2}{n\pi}(\cos n\pi - 1)$$

Jadi

$$b_n = -\frac{2}{n\pi}(\cos n\pi - 1)$$

Untuk $n = 1$ maka $b_1 = -\frac{2}{1\pi}(\cos 1\pi - 1) = -\frac{2}{\pi}(-1-1) = -\frac{-4}{\pi} = \frac{4}{\pi}$
Untuk $n = 2$ maka $b_2 = -\frac{2}{2\pi}(\cos 2\pi - 1) = -\frac{2}{2\pi}(1-1) = 0$
Untuk $n = 3$ maka $b_3 = -\frac{2}{3\pi}(\cos 3\pi - 1) = -\frac{2}{3\pi}(-1-1) = -\frac{-4}{3\pi} = \frac{4}{3\pi}$
Untuk $n = 4$ maka $b_2 = \frac{2}{4\pi}(\cos 4\pi - 1) = \frac{2}{4\pi}(1-1) = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x$$

$$= b_1 \sin\frac{1\pi}{2}x + b_2 \sin\frac{2\pi}{2}x + b_3 \sin\frac{3\pi}{2}x + b_4 \sin\frac{4\pi}{2}x + \cdots$$

$$= \frac{4}{\pi}\sin\frac{\pi}{2}x + 0.\sin\frac{2\pi}{2}x + \frac{4}{3\pi}\sin\frac{3\pi}{2}x + 0.\sin\frac{4\pi}{2}x + \cdots$$

$$= \frac{4}{\pi}\sin\frac{\pi}{2}x + \frac{4}{3\pi}\sin\frac{3\pi}{2}x + \cdots$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2}x$$