Pertemuan 5 KALKULUS III

INTEGRAL RANGKAP

Integral Tentu

Misalkan

$\int f(x)\,dx = F(x) + C$

$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

Contoh :

$\int_1^2 x + 5\,dx = \frac{1}{2}x^2 + 5x\Big|_1^2 = \left(\frac{1}{2}2^2 + 5.2\right) - \left(\frac{1}{2}1^2 + 5.1\right) = \left(\frac{1}{2}.4 + 10\right) - \left(\frac{1}{2}.1 + 5\right)$

$= \left(\frac{4}{2} + 10\right) - \left(\frac{1}{2} + 5\right) = \left(\frac{4}{2} - \frac{1}{2}\right) + (10 - 5) = \left(\frac{3}{2} + 5\right)$

$= \left(\frac{3}{2} + 5\right) = \left(\frac{3}{2} + \frac{10}{2}\right) = \frac{13}{2}$

Kerjakan

1. $\int_0^2 3x - 1\,dx = \frac{3}{2}x^2 - x\Big|_{..}^{..} = \left(\frac{3}{2}\ldots^2 - \ldots\right) - \left(\frac{\ldots}{2}0^2 - \ldots\right) = \cdots$
2. $\int_1^3 x^2 - x\,dx = \frac{1}{3}x^{\cdots} - \frac{\cdots}{\cdots}x^{\cdots}\Big|_{..}^{..} = \cdots$
3. $\int_{-1}^3 5x^2 + 2x\,dx = \cdots$

Integral lipat 2

$\iint_R f(x,y)dA = \iint_R f(x,y)dxdy = \iint_R f(x,y)dydx$

Dengan R daerah berbentuk persegi panjang sederhana di bidang XY (sejajar dengan sumbu koordinat)
$R = \{(x,y): a \le x \le b, c \le y \le d\}$

$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_a^b F_1(x,y)\Big|_c^d dx = \int_a^b f_2(x)dx = F_2(x)\Big|_a^b$

Dengan $F_1(x,y)$ adalah anti turunan/integral $f(x,y)$ terhadap y, x ditahan agar tetap konstan.

Contoh :

1. $\int_0^1 \int_0^2 x^2 y\,dydx = \int_0^1 x^2\,\frac{1}{2}y^2\Big|_0^2 dx$ diintegralkan terhadap y (x ditahan tetap konstan)

$= \int_0^1 ((x^2.\frac{1}{2}2^2) - (x^2.\frac{1}{2}0^2))\,dx$ variabel y diganti dengan nilai batas y

$= \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \left(\frac{2}{3}1^3 - 0\right) = \frac{2}{3}$ $\int_0^1 \int_0^2 x^2 + y\,dydx$

$= \int_0^1 x^2 y + \frac{1}{2}y^2\Big|_0^2 dx$ diintegralkan terhadap y (x ditahan tetap konstan)

$= \int_0^1 (x^2.2 + \frac{1}{2}.2^2) - (x^2.0 + \frac{1}{2}0^2)\,dx$ variabel y diganti nilai batas y

$= \int_0^1 (2x^2 + 2)\,dx = \frac{2}{3}x^3 + 2x\Big|_0^1 = \left(\frac{2}{3}1^3 + 2.1\right) - \left(\frac{2}{3}0^3 + 2.0\right)$

$= \frac{2}{3} + 2 = \frac{2+6}{3} = \frac{8}{3}$

3. $\int_0^{\pi}\int_1^2 x^2 siny\ dxdy = \int_0^{\pi} \frac{1}{3}x^3 siny|_1^2\, dy$

diintegralkan terhadap x (y ditahan ttp konstan)

$= \int_0^{\pi}\left(\frac{1}{3}.2^3 siny\right) - \left(\frac{1}{3}.1^3 siny\right)\ dy$ variabel x diganti nilai batas x

$= \int_0^{\pi}\left(\frac{8}{3}siny\right) - \left(\frac{1}{3}siny\right)\ dy = \int_0^{\pi} \frac{7}{3} siny\ dy$

$= \frac{7}{3}.-cosy\ \Big|_0^{\pi} = -\frac{7}{3}cosy\ \Big|_0^{\pi} = \left(-\frac{7}{3}cos\pi\right) - \left(-\frac{7}{3}cos0\right)$

$= \left(-\frac{7}{3}.-1\right) - \left(-\frac{7}{3}.1\right) = \frac{7}{3} + \frac{7}{3} = \frac{14}{3}$

4. $\int_1^2\int_0^{\frac{\pi}{2}} r.cos\theta\ d\theta dr = \int_1^2 r.sin\theta\Big|_0^{\frac{\pi}{2}}\ dr$diintegralkan terhadap Θ (r ditahan ttp konstan)

$= \int_1^2\ \left(r.sin\frac{\pi}{2}\right) - (r.sin0)\ dr$ variabel Θ diganti nilai batas Θ

$= \int_1^2\ (r.1) - (r.0)\ dr$

$= \int_1^2 r\ dr = \frac{1}{2}r^2\Big|_1^2 = (\frac{1}{2}.2^2) - \left(\frac{1}{2}.1^2\right) = 2 - \frac{1}{2} = \frac{3}{2}$

Integral lipat 3

$\iiint_B\ dV$

Contoh :

$\int_{-1}^1\int_1^2\int_0^3 x^3 - 2y^2z\ dydzdx$ diintegralkan terhadap y (x dan z ditahan ttp konstan)

$\int_{-1}^1\int_1^2\int_0^3 x^3 - 2y^2z\ dydzdx = \int_{-1}^1\int_1^2 x^3y - \frac{2}{3}y^3z\Big|_0^3\ dzdx$

$= \int_{-1}^1\int_1^2\left(x^3.3 - \frac{2}{3}3^3z\right) - \left(x^3.0 - \frac{2}{3}0^3z\right)\ dzdx$

$= \int_{-1}^1\int_1^2(3x^3 - 18z)\ dzdx$

$= \int_{-1}^1 3x^3z - 9z^2\Big|_1^2\ dx$

$= \int_{-1}^1(3x^3.2 - 9.2^2) - (3x^3.1 - 9.1^2)\ dx$

$= \int_{-1}^1(6x^3 - 36) - (3x^3 - 9)\ dx$

$= \int_{-1}^1(3x^3 - 27)\ dx$

$= \frac{3}{4}x^4 - 27x\Big|_{-1}^1$

$= (\frac{3}{4}1^4 - 27.1) - (\frac{3}{4}.(-1)^4 - 27.(-1) = -54$

Latihan

1. $\int_0^1\int_1^2 2xy\ dydx = \int_0^1 2x.\frac{\cdots}{\cdots}y^{\cdots}\Big|_{\cdots}^2\ dx = \cdots$
2. $\int_0^1\int_0^2 x + y^2\ dxdy = \int_0^1 \frac{1}{2}\ldots^2 + y^2.\ldots\Big|_0^2\ dx = \cdots$
3. $\int_0^{\pi}\int_1^2 3x^2 cosy\ dxdy = \cdots$
4. $\int_1^2\int_0^{\frac{\pi}{2}} 2r\ cos\theta + r\ sin\ \theta\ d\theta dr = \cdots$
5. $\int_{-1}^1\int_1^2\int_0^3 xy^2z\ dzdydx = \cdots$

Integral lipat atas daerah persegi panjang R dengan daerah persegi panjang R terbagi menjadi

Misalkan diberikan daerah R dengan

$$R_1 = \{(x,y): 0 \le x \le 1, 0 \le y \le 1\}$$
$$R_2 = \{(x,y): 1 \le x \le 3, 0 \le y \le 1\}$$

dan

$$f(x,y) = \begin{cases} 3 & ;\ 0 \le x \le 1, 0 \le y \le 1 \\ 2 & ;\ 1 \le x \le 3, 0 \le y \le 1 \end{cases}$$

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

maka

$$\iint_R f(x,y)dA = \int_0^1 \int_0^3 f(x,y)dx\,dy = \int_0^1 \int_0^1 3\,dx\,dy + \int_0^1 \int_1^3 2\,dx\,dy \quad \int_0^1 \int_0^3 f(x,y)dx\,dy = \int_0^1 3x\Big|_0^1 dy + \int_0^1 2x\Big|_1^3 dy$$

$$\begin{aligned} &= \int_0^1 (3.1 - 3.0)\,dy + \int_0^1 (2.3 - 2.1)\,dy \\ &= \int_0^1 (3 - 0)\,dy + \int_0^1 (6 - 2)\,dy \\ &= \int_0^1 3\,dy + \int_0^1 4\,dy = 3y\Big|_0^1 + 4y\Big|_0^1 \\ &= (3.1 - 3.0) + (4.1 - 4.0) = 3 + 4 = 7 \end{aligned}$$

Latihan :

1. Diberikan

   $R_1 = \{(x,y): -1 \le x \le 1, 1 \le y \le 2\}$
   $R_2 = \{(x,y): \quad 1 \le x \le 3, 1 \le y \le 2\}$

   tentukan

   $$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$
   $$\iint_R dA = \int \quad \int \quad dx\,dy + \int \quad \int \quad dx\,dy$$
   $$= \int \quad x\Big|_{\cdots}^{\cdots} dy + \cdots = \cdots$$

2. Diberikan

   $R_1 = \{(x,y): 1 \le x \le 3, 1 \le y \le 2\}$
   $R_2 = \{(x,y): 1 \le x \le 3, 2 \le y \le 3\}$

   tentukan

   $$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$
   $$\iint_R dA = \int \quad \int \quad dy\,dx + \int \quad \int \quad dy\,dx = \cdots$$

3. Diberikan

   $R_1 = \{(x,y): 0 \le x \le 2, 1 \le y \le 2\}$
   $R_2 = \{(x,y): 0 \le x \le 2, 2 \le y \le 3\}$

   Dengan

   $$f(x,y) = \begin{cases} x & ;\ 0 \le x \le 2, 0 \le y \le 1 \\ x^2 & ;\ 0 \le x \le 2, 1 \le y \le 2 \end{cases}$$

   tentukan

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

Integral lipat atas daerah bukan persegi panjang

$\iint_S f(x,y)dA = \iint_S f(x,y)dxdy = \iint_S f(x,y)dydx$

Dengan S adalah himpunan x sederhana atau S adalah himpunan y sederhana

- S himpunan x sederhana,
  $S = \{(x,y): \phi_1 \le x \le \phi_2, c \le y \le d\}$
  Integral lipatnya : $\int_c^d \int_{\phi_1}^{\phi_2} f(x,y)dxdy$
- S adalah himpunan y sederhana
  $S = \{(x,y): a \le x \le b, \varphi_1 \le y \le \varphi_2\}$
  Integral lipatnya : $\int_a^b \int_{\varphi_1}^{\varphi_2} f(x,y)dydx$

Contoh:

1. $f(x,y) = x^2y$ dengan $S = \{(x,y): 0 \le x \le 1, 0 \le y \le x\}$ himpunan y sederhana, maka integral lipatnya

   $\int_0^1 \int_0^x x^2y\, dydx = \int_0^1 x^2.\frac{1}{2}y^2\Big|_0^x\, dx$ diintegralkan terhadap y (x ditahan tetap konstan)

   $= \int_0^1 (x^2.\frac{1}{2}x^2) - \left(x^2.\frac{1}{2}0^2\right)dx$ variabel y diganti dengan nilai batas y

   $= \int_0^1 \frac{1}{2}x^4 - 0\, dx = \int_0^1 \frac{1}{2}x^4\, dx = \frac{1}{2}\frac{1}{5}x^5\Big|_0^1 = \frac{1}{10}x^5\Big|_0^1 = \frac{1}{10}$

2. $(x,y) = x^2 + y$ dengan $S = \{(x,y): 1 \le x \le 2,\ x^2 \le y \le x+2\}$ himpunan y sederhana, maka integral lipatnya

   $\int_1^2 \int_{x^2}^{x+2} x^2 + y\, dydx = \int_1^2 x^2y + \frac{1}{2}y^2\Big|_{x^2}^{x+2}\, dx$ diintegralkan terhdp y (x ditahan ttp konstan)

   $= \int_1^2 \{(x^2(x+2) + \frac{1}{2}(x+2)^2) - \left(x^2x + \frac{1}{2}x^2\right)\}\, dx$

   var y diganti dg nilai batas y

   $= \int_1^2 \{\ x^3 + 2x^2 + \frac{1}{2}(x^2 + 4x + 4) - (x^3 + \frac{1}{2}x^2)\}\, dx$

   $= \int_1^2 \{\ x^3 + 2x^2 + \frac{1}{2}x^2 + 2x + 2 - x^3 - \frac{1}{2}x^2\}\, dx$

   $= \int_1^2 \{\ \cancel{x^3} + 2x^2 + \cancel{\frac{1}{2}x^2} + 2x + 2 - \cancel{x^3} - \cancel{\frac{1}{2}x^2}\}\, dx$

   $= \int_1^2 \{\ 2x^2 + 2x + 2\}\, dx = \frac{2}{3}x^3 + x^2 + 2x\Big|_1^2$

   $= \left(\frac{2}{3}2^3 + 2^2 + 2.2\right) - \left(\frac{2}{3}1^3 + 1^2 + 2.1\right) = \frac{2}{3}.8 + 4 + 4 - \frac{2}{3} - 1 - 2$

   $= \frac{14}{3} + 5 = \frac{29}{3}$

3. $f(x,y) = 1$ dengan $S = \{(x,y): y^{\frac{1}{3}} \leq x \leq y^{\frac{1}{2}}, 0 \leq y \leq 1\}$ himpunan x sederhana, maka integral lipatnya

$\int_0^1 \int_{y^{\frac{1}{3}}}^{y^{\frac{1}{2}}} 1\, dxdy = \int_0^1 x \,\Big|_{y^{\frac{1}{3}}}^{y^{\frac{1}{2}}} dy$ diintegralkan terhadap x (y ditahan tetap konstan)

$= \int_0^1 (y^{\frac{1}{2}} - y^{\frac{1}{3}})dy$ variabel x diganti dengan nilai batas x

$= (\frac{2}{3}y^{\frac{3}{2}} - \frac{3}{4}y^{\frac{4}{3}})\Big|_0^1 = (\frac{2}{3}.1^{\frac{3}{2}} - \frac{3}{4}.1^{\frac{4}{3}}) - (\frac{2}{3}.0^{\frac{3}{2}} - \frac{3}{4}.0^{\frac{4}{3}})$

$= \frac{2}{3} - \frac{3}{4} = \frac{2.4-3.3}{12} = -\frac{1}{12}$

4. $f(r,\theta) = e^{\cos\theta}$ dengan $S = \{(r,\theta): 0 \leq r \leq \sin\theta, 0 \leq \theta \leq \pi\}$ himpunan r sederhana, maka integral lipatnya

$\int_0^\pi \int_0^{\sin\theta} e^{\cos\theta}\, drd\theta = \int_0^\pi e^{\cos\theta} r\Big|_0^{\sin\theta} d\theta$

diintegralkan terhdp r (ɵ ditahan ttp konstan)

$= \int_0^\pi (e^{\cos\theta}.\sin\theta - e^{\cos\theta}.0)d\theta$ variabel x diganti dg nilai batas x

$= \int_0^\pi (e^{\cos\theta}.\sin\theta)d\theta$

$= -e^{\cos\theta}\Big|_0^\pi = (-e^{\cos\pi}) - (-e^{\cos o}) = -e^{-1} - (-1) = 1 - \frac{1}{e}$

Catt :

$\int e^{\cos\theta}.\sin\theta\ d\theta$: misalkan $u = \cos\theta$ dan $\frac{du}{d\theta} = -\sin\theta$

$d\theta = \frac{du}{-sin\ \theta}$

$\int e^{\cos\theta}.\sin\theta\, d\theta = \int e^u.\cancel{\sin\theta}.\frac{du}{-\cancel{sin\ \theta}} = -\int e^u\, du = -e^u + c = -e^{\cos\theta} + c$

5. $f(x,y) = 1$ dengan $S = \{(x,y): 0 \leq x \leq 1, x^2 \leq y \leq x\}$ himpunan x sederhana, maka integral lipatnya

$\int_0^1 \int_{x^2}^{x} 1\, dydx = \int_0^1 y\,\Big|_{x^2}^{x} dx$ diintegralkan terhadap x (y ditahan tetap konstan)

$= \int_0^1 (x - x^2)dx$ variabel x diganti dengan nilai batas x

$= (\frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_0^1 = (\frac{1}{2}.1^2 - \frac{1}{3}.1^3) - (\frac{1}{2}.0^2 - \frac{1}{3}.0^3)$

$= \frac{1}{2} - \frac{1}{3} = \frac{1.3-1.2}{6} = \frac{1}{6}$

DAFTAR PUSTAKA : Purcell, E.J. & Vanberg, D.E. (1992). *Kalkulus dan Deometri Analitis*, Jilid 2. Jakarta: Erlangga.